

Robust Adaptive Control of a Cantilevered Flexible Structure with Spatiotemporally Varying Coefficients and Bounded Disturbance*

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In this paper, a robust model reference adaptive control of a cantilevered flexible structure with unknown spatiotemporally varying coefficients and disturbance is investigated. Any mechanically flexible manipulators/structures are inherently distributed parameter systems whose dynamics are described by partial, rather than ordinary, differential equations. Robust adaptive control laws are derived by the Lyapunov redesign method on an infinite dimensional Hilbert space. Under the assumption that disturbances are uniformly bounded, the proposed robust adaptive scheme guarantees the boundedness of all signals in the closed loop system and the convergence of the state error near to zero. With an additional persistence of excitation condition, the parameter estimation errors are shown to converge near to zero as well.

Key Words: Robust Model Reference Adaptive Control, Cantilevered Flexible Structure, Stability, Lyapunov Method, Persistence of Excitation

1. Introduction

Almost every engineering system will exhibit some distributed parameter behavior if one looks at its dynamics in great detail. Consequently, the dynamical behavior of such a distributed parameter system (DPS) would need to be modeled by partial, rather than ordinary, differential equations (PDEs) to be correctly represented. Of course, in many cases, such detail is not necessary for the successful operation of the system, and a lumped parameter (ordinary differential equation, ODE) model is satisfactory. Nevertheless, a large number of current and newly proposed systems, such as industrial processes and mechanically flexible robot manipulators and flexible spacecraft and satellites, are so thoroughly distributed parameter in nature that it is impossible to ignore this in modeling and control⁽¹⁾⁻⁽⁴⁾.

Such DPSs are described by operator equations on an

infinite dimensional Hilbert (or Banach) space. The analysis of DPSs then makes use of the theory of semigroup on an infinite dimensional state space. The infinite dimensional approach will yield results that can be used effectively in large-scale finite dimensional systems as well. One very important consideration in large-scale or distributed parameter systems is to avoid the dependence on precise knowledge of the total system dimension and the full system parameters, especially those residual parameters that are not used in the synthesis of the controller. This infinite dimensional approach can eliminate the uncertainty on system dimension and the spillover problems on residual data.

The mathematical models of physical plants that control engineers formulate to design a control system normally contain some uncertainty. This is due to imperfect knowledge on the system parameters and/or disturbances. The unstructured uncertainty is due to unmodeled dynamics, for instance, neglected frictions, neglected high order dynamics, etc., and may also arise from linear approximations along different motions over a wide range of operating conditions. In flexible systems, not only the geometry of the structure but also physical properties such as the density, stiffness, Poisson ratio, and damping coefficients may change. Indeed, many of these spatiotemporally varying parameters are unknown even if the material itself is homogeneous and the structure is uniform.

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Thus, the control problem of flexible structures provides challenging issues including parameter estimation, uncertainty quantification, and robustness. Compared to the finite dimensional case, the adaptive control of infinite dimensional systems is not well developed and has only recently been studied^{(5)–(10)}.

In this paper, a robust model reference adaptive control (MRAC) of a cantilevered flexible beam with unknown spatiotemporally varying coefficients and disturbance is developed. The objective of an MRAC scheme is to determine a feedback control law which forces the state of the plant to asymptotically track the state of a given reference model. At the same time, the unknown parameters in the plant model are estimated and used to update the control law.

Possible applications of the robust MRAC scheme considered in this paper are shape control, tracking control, and vibration control of various flexible structures, e.g., very long arms needed for accessing hostile environments (nuclear sites, underground waste deposits, deep sea, space, etc.) or automated crane devices for building construction. Shape control involves activating the structure in order to achieve a certain desired shape specified by the user. Applications range from controlling the shape of flexible arms such as a micro manipulator⁽¹¹⁾ to large flexible space structures⁽¹²⁾ that smart actuators integrated within the structure produce small in-plane deflections that can in turn produce large out-of-plane deformations. Such smart structures incorporating adaptive materials, which can then be used as distributed actuators or sensors, into the main host structure are receiving much attention as an advanced control technology⁽¹³⁾. Studies on the shape control using distributed actuators/sensors include Chee et al.⁽¹⁴⁾ for plates, and Chandrashekhara and Varadarajan⁽¹⁵⁾ for beams.

The present paper makes the following contributions: A cantilevered flexible beam in the frame of robust MRAC is treated in this paper. To the authors' best knowledge this paper is the first treatment of an infinite dimensional system with unknown spatiotemporally varying parameters and additive unknown spatiotemporally varying disturbance in the frame of robust MRAC. The unknown time-varying parameters are not required to be slow, which can be allowed to vary arbitrarily fast. The well-posedness of the closed loop system is established via the theory of infinite dimensional evolution equations. Using an appropriate Lyapunov function, the asymptotic convergence of the tracking error near to zero is established. With an additional assumption of persistence of excitation, the convergence of parameter estimation errors near to zero is established as well.

The rest of this article is organized as follows: In section 2, the dynamic equations (PDEs) of the cantilevered flexible beam and the reference model are formulated. The

control law and the adaptation laws are also presented; the well-posedness of the coupled nonlinear system consisting of the state error equation and the adaptation laws is shown in section 3; the convergence of the tracking and parameter estimation errors near to zero is presented in section 4, followed by the conclusions in section 5.

2. Problem Formulation: Robust MRAC

In this paper, as shown schematically in Fig. 1, a cantilevered flexible beam of length l fixed at $x = 0$ and free at $x = l$, with viscous damping, is considered. In many cases, such a simple model retains the essential features of more complicated flexible robots/structures including micro-electro-mechanical systems (MEMS). Neglecting the effect of gravity and rotatory inertia of the beam cross-sections and using the Euler-Bernoulli beam model, the following equations of motion for the one dimensional Euler-Bernoulli beam with spatiotemporally varying coefficients and disturbance⁽¹⁶⁾ is derived:

$$\begin{aligned} & \rho(x,t)u_{tt}(x,t) \\ & + \frac{\partial^2}{\partial x^2} \left[EI(x,t) \left(\frac{\partial^2 u(x,t)}{\partial x^2} + \alpha_c(x,t) \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \right) \right] \\ & = f(x,t) + d_0(x,t), \\ & 0 < x < l, \quad t > 0, \\ & u(0,t) = \frac{\partial u(x,t)}{\partial x} \Big|_{x=0} = 0, \\ & EI \left(\frac{\partial^2 u(x,t)}{\partial x^2} + \alpha_c \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \right) \Big|_{x=l} \\ & = \frac{\partial}{\partial x} \left[EI \left(\frac{\partial^2 u(x,t)}{\partial x^2} + \alpha_c \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \right) \right] \Big|_{x=l} = 0, \\ & u(x,0) = u_0(x), \quad u_t(x,0) = u_{t0}(x) \end{aligned} \quad (1)$$

where $u(x,t)$ is the transverse-displacement (and also is the observed distributed state), $\rho(x,t)$ is the mass per unit length, $EI(x,t)$ is the flexural stiffness, $\alpha_c(x,t)$ is the stiffness proportionality factor defined for Rayleigh damping and denotes the internal resistance opposing the strain velocity, $f(x,t)$ is the control input force, $d_0(x,t)$ is the inaccessible external disturbance, and $u_0(x)$ and $u_{t0}(x)$ are the initial conditions, $u_t = \partial u / \partial t$ and $u_{tt} = \partial^2 u / \partial t^2$. The

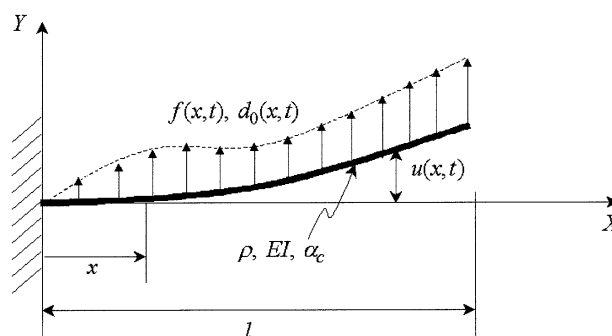


Fig. 1 Flexible beam deflected due to loadings.

boundary conditions indicate that the beam is fixed at $x = 0$ and free at $x = l$. Assume the physical constraints $0 < \vartheta_1 \leq \rho(x, t) \leq \vartheta_2$, $\vartheta_1 \leq EI(x, t)$, and $\vartheta_1 \leq \alpha_c EI(x, t)$ for all $x \in [0, l]$ and $t \geq 0$ with a priori known constants ϑ_i , $i = 1, 2$. Note that $d_0(x, t)$ is an unknown spatiotemporally varying function but uniformly bounded.

A typical problem related to this model would be to control the system while the coefficients are unknown, i.e., $\rho(x)$, $EI(x)$, and $\alpha_c EI(x)$ are unknown, which may vary arbitrarily with position x along the span l . That is, the parameters may be piecewise values due to the presence and differing material properties of the bonding layer and patches of distributed actuators and sensors. It is then forced to differentiate discontinuous functions when considering the strong form of plant equation (1). To avoid the difficulty as well as lower smoothness requirements for approximating elements, the system (1) in weak form is considered.

To convert (1) into the weak form, both sides of (1) are multiplied by a sufficiently smooth test function φ and are integrated by parts. Assuming that φ satisfies the boundary conditions $\varphi(x) = \partial\varphi(x)/\partial x = 0$ at $x = 0$ and $\varphi \in C^\infty(0, l)$, the weak form of (1) is

$$\int_{\Gamma} \left(\rho(x, t) u_{tt}(x, t) + \frac{\partial^2}{\partial x^2} \left(EI(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + \alpha_c EI(x, t) \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \right) - f(x, t) - d_0(x, t) \right) \varphi(x) dx = 0 \tag{2}$$

where $\Gamma = [0, l]$. The integration of the second and third terms by part twice yields:

$$\begin{aligned} & \int_{\Gamma} \rho(x, t) u_{tt}(x, t) \varphi(x) dx + \int_{\Gamma} EI(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} \frac{\partial^2 \varphi(x)}{\partial x^2} dx \\ & + \int_{\Gamma} \alpha_c EI(x, t) \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \frac{\partial^2 \varphi(x)}{\partial x^2} dx \\ & - \int_{\Gamma} (f(x, t) + d_0(x, t)) \varphi(x) dx \\ & + \left[\frac{\partial}{\partial x} \left(EI(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} \right) \varphi(x) - EI(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} \frac{\partial \varphi(x)}{\partial x} \right]_0^l \\ & + \left[\frac{\partial}{\partial x} \left(\alpha_c EI(x, t) \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \right) \varphi(x) - \alpha_c EI(x, t) \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \frac{\partial \varphi(x)}{\partial x} \right]_0^l = 0. \end{aligned} \tag{3}$$

Now using the boundary conditions, the following weak form is derived:

$$\int_{\Gamma} \rho(x, t) u_{tt}(x, t) \varphi(x) dx + \int_{\Gamma} EI(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} \frac{\partial^2 \varphi(x)}{\partial x^2} dx + \int_{\Gamma} \alpha_c EI(x, t) \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \frac{\partial^2 \varphi(x)}{\partial x^2} dx$$

$$= \int_{\Gamma} f(x, t) \varphi(x) dx + \int_{\Gamma} d_0(x, t) \varphi(x) dx. \tag{4}$$

Then, (4) can be rewritten as

$$\begin{aligned} & \int_{\Gamma} u_{tt}(x, t) \varphi(x) dx + \int_{\Gamma} q_1(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} \frac{\partial^2 \varphi(x)}{\partial x^2} dx \\ & + \int_{\Gamma} q_2(x, t) \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \frac{\partial^2 \varphi(x)}{\partial x^2} dx \\ & = \int_{\Gamma} q_3(x, t) f(x, t) \varphi(x) dx + \int_{\Gamma} d(x, t) \varphi(x) dx, \end{aligned} \tag{5}$$

where $q_1(x, t)$, $q_2(x, t)$, $q_3(x, t)$, and $d(x, t)$ denote $\rho^{-1}(x, t)EI(x, t)$, $\rho^{-1}(x, t)\alpha_c EI(x, t)$, $\rho^{-1}(x, t)$, and $\rho^{-1}(x, t)d_0(x, t)$, respectively^{(9), (17)}. The coefficients q_i , $i = 1, 2, 3$, are unknown. It is pointed out that in the weak form some derivatives have been transferred from the beam moments to the test function. In this paper it is assumed that the system state $u(x, t)$ can be measured at all points of $x \in \Gamma$ and $t \geq 0$.

To pose the MRAC problem, adequate function spaces are now introduced. Let H and V be the Hilbert spaces given by $H = L^2(\Gamma)$ and $H = H_L^2(\Gamma)$, which are defined as follows:

$$\begin{aligned} L^2(\Gamma) &= \left\{ \eta: [0, l] \rightarrow R \mid \int_{\Gamma} \eta^2 dx < \infty \right\}, \\ H_L^2(\Gamma) &= \left\{ \eta \in L^2(\Gamma) \mid \frac{\partial \eta}{\partial x}, \frac{\partial^2 \eta}{\partial x^2} \in L^2(\Gamma), \right. \\ & \left. \text{and } \eta(x) = \frac{\partial \eta(x)}{\partial x} = 0 \text{ at } x = 0 \right\}, \end{aligned} \tag{6}$$

where the subscript L indicates that only left side is supported. The inner products in H and V are defined, respectively, as

$$\langle \psi(x), \varphi(x) \rangle_H = \int_{\Gamma} \psi(x) \varphi(x) dx, \tag{7}$$

$$\langle \psi(x), \varphi(x) \rangle_V = \int_{\Gamma} q_1^*(x) \frac{\partial^2 \psi(x)}{\partial x^2} \frac{\partial^2 \varphi(x)}{\partial x^2} dx, \tag{8}$$

where $q_1^*(x) > 0$ is a reference model parameter and is specified in (13), and the corresponding induced norms are denoted by $\|\cdot\|$ and $\|\cdot\|_V$, respectively. Since V is densely and continuously embedded in H , the Hilbert spaces H and V form a Gelfand triple⁽¹⁸⁾

$$V \hookrightarrow H \cong H^* \hookrightarrow V, \tag{9}$$

where H^* and V^* denote the topological dual spaces of H and V , respectively, and denote embedding. As usual, the Hilbert space H^* is identified by H . Since the embeddings in (9) are dense and continuous, the following is then satisfied:

$$\|\psi\| \leq K \|\psi\|_V, \quad \psi \in V, \tag{10}$$

for some positive (embedding) constant K . Let $q_i \in Q$, $i = 1, 2, 3$, where Q is a compact subset of $L^2(\Gamma)$ and is a real Hilbert space (henceforth the parameter space) with inner product $\langle \cdot, \cdot \rangle_Q$, and corresponding norm $\|\cdot\|_Q$. Note that the parameter space Q is appropriately chosen according

to the bonding layer and patches of distributed actuators and sensors⁽¹⁹⁾.

For $q \in Q$ and $\psi, \varphi \in V$ sesquilinear forms $\sigma_i(q; \cdot, \cdot): V \times V \rightarrow R, i = 1, 2$, are defined on V as

$$\sigma_i(q; \psi, \varphi) = \int_{\Gamma} q(x) \frac{\partial^2 \psi(x)}{\partial x^2} \frac{\partial^2 \varphi(x)}{\partial x^2} dx, \quad (11)$$

where $q(x) > 0$. The sesquilinear forms $\sigma_i(q; \cdot, \cdot), i = 1, 2$, satisfy various continuity, symmetricity, coercivity, and linearity conditions, i.e.,

$$(A1) \quad |\sigma_i(q; \psi, \varphi)| \leq k_1(q) \|\psi\|_V \|\varphi\|_V, \quad \text{for some } k_1(q) > 0 \quad (\text{boundedness}),$$

$$(A2) \quad \sigma_i(q; \psi, \psi) \geq k_2(q) \|\psi\|_V^2, \quad \text{for some } k_2(q) > 0 \quad (\text{coercivity}),$$

$$(A3) \quad \sigma_i(q; \psi, \varphi) = \sigma_i(q; \varphi, \psi) \quad (\text{symmetricity}),$$

$$(A4) \quad \text{The map } q \rightarrow \sigma_i(q; \psi, \varphi) \text{ from } Q \text{ into } R \text{ is linear} \quad (\text{linearity}),$$

for $\varphi, \psi \in V$. While the symmetricity (A3) and linearity (A4) follow directly from (11), the boundedness (A1) results from the Schwarz's inequality for inner products and the coercivity (A2) follows from the fact that there exists a $k_2(q) > 0$ such that

$$\begin{aligned} \sigma_i(q; \psi, \psi) &= \int_{\Gamma} q(x) \left(\frac{\partial^2 \psi(x)}{\partial x^2} \right)^2 dx \\ &\geq k_2(q) \int_{\Gamma} q_1^*(x) \left(\frac{\partial^2 \psi(x)}{\partial x^2} \right)^2 dx = k_2(q) \|\psi\|_V^2, \end{aligned}$$

for $i = 1, 2$ and $\psi \in V$.

With these definitions, (5) with (1) can be rewritten as

$$\begin{aligned} \langle u_{tt}, \varphi \rangle + \sigma_1(q_1; u, \varphi) + \sigma_2(q_2; u_t, \varphi) &= \langle q_3 f, \varphi \rangle + \langle d, \varphi \rangle, \\ 0 < x < l, \quad t > 0, \\ u(0, t) = \frac{\partial u(x, t)}{\partial x} \Big|_{x=0} &= 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) &= u_{t0}(x), \end{aligned} \quad (12)$$

where $u_0(x) \in V, u_{t0}(x) \in H$, and the notation $\langle \cdot, \cdot \rangle$ denotes the usual duality product obtained as the extension by continuity of the H -inner product from $H \times V$ to $V^* \times V$ ⁽¹⁸⁾. The control input force $f(x, t)$ and the disturbance $d(x, t)$ are assumed to satisfy $f, d \in L^2(0, T; V^*)$ for all $T > 0$. Note that $\|d(x, t)\|$ is assumed to be uniformly bounded by $\mu_d(t)$, i.e., $\mu_d(t) \geq \|d(x, t)\|$, where $\mu_d(t)$ is a time-varying function that is unknown.

The robust MRAC problem for plant (12), in the presence of unknown parameters $q_i, i = 1, 2, 3$, and d , is now to find the control input f in feedback form that forces the state u to track a reference signal v . The reference signal v is generated through a reference model defined by

$$\begin{aligned} \langle v_{tt}, \varphi \rangle + \sigma_1(q_1^*; v, \varphi) + \sigma_2(q_2^*; v_t, \varphi) &= \langle q_3^* g, \varphi \rangle, \\ 0 < x < l, \quad t > 0, \\ v(0, t) = \frac{\partial v(x, t)}{\partial x} \Big|_{x=0} &= 0, \end{aligned}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_{t0}(x), \quad (13)$$

for $\varphi \in V$, where $v_0(x) \in V, v_{t0}(x) \in H$, and $0 < q_{i0}^* \leq q_i^*(x), i = 1, 2, 3$, with constants q_{i0}^* 's. The reference model parameters q_i^* 's are sufficiently smooth and chosen so that the response (v, v_t) can have the desired characteristics, and the input reference signal $g(x, t)$ is assumed to satisfy $g \in L^2(0, T; V^*)$ for all $T > 0$.

Let us define the state error e as

$$e(t, x) = u(t, x) - v(t, x). \quad (14)$$

The control objective for MRAC is to find a bounded control law, f , which drives u to v and u_t to v_t , asymptotically. More precisely, f is chosen to achieve

$$\lim_{t \rightarrow \infty} \|e_t\| = \lim_{t \rightarrow \infty} \|e\|_V \approx 0 \quad (15)$$

with all the signals in the closed loop bounded.

Consider the nominal control input, f^* , as follows:

$$\langle f^*, \varphi \rangle = \sigma_1(\theta_1^*; u, \varphi) + \sigma_2(\theta_2^*; u_t, \varphi) + \langle \theta_3^* g, \varphi \rangle, \quad (16)$$

where $\theta_1^* = q_3^{-1}(q_1 - q_1^*), \theta_2^* = q_3^{-1}(q_2 - q_2^*),$ and $\theta_3^* = q_3^{-1} q_3^*$. If the disturbance $d(x, t) = 0$, then, by substituting the nominal control input into (12), it is seen that (12) coincides with (13), i.e., the plant equation and the reference model equation become identical. But, because $q_i, i = 1, 2, 3$, are unknown, the values of θ_i^* in (16) are not known. Hence, in the case of the plant (12) including unknown time-varying coefficients and disturbance, a novel adaptive control law should be introduced. The main idea about the control law is to consider the worst case of the uncertainties in the form of possible bounds. Based upon the worst case, the following control algorithm is proposed.

$$\begin{aligned} f = \frac{\partial^2}{\partial x^2} \left(\hat{\theta}_1(t) \frac{\partial^2 u(x, t)}{\partial x^2} + \hat{\theta}_2(t) \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \right) \\ + (\hat{\theta}_3(t) g - \mu_0 e_t - \mu e + f_c), \end{aligned}$$

or

$$\begin{aligned} \langle f, \varphi \rangle = \sigma_1(\hat{\theta}_1(t); u, \varphi) + \sigma_2(\hat{\theta}_2(t); u_t, \varphi) + \langle \hat{\theta}_3(t) g, \varphi \rangle \\ - \langle \mu_0 e_t, \varphi \rangle - \langle \mu e, \varphi \rangle + \langle f_c, \varphi \rangle, \end{aligned} \quad (17)$$

for each $t > 0$, where $\mu_0 > 1$ and $\mu > 0$ are introduced to guarantee the convergence of the state error, see section 3, and $\hat{\theta}_i(t) \in Q, i = 1, 2, 3$, denote adaptively updated estimates for θ_i^* , respectively. The additional term $f_c(x, t)$ is regarded as a new input signal to be determined based on robust control strategy. The additional input $f_c(x, t)$ is given by

$$f_c(x, t) = - \frac{\hat{\mu}_d^2(t)}{\hat{\mu}_d(t) \|e(x, t) + e_t(x, t)\| + \varepsilon_d} (e(x, t) + e_t(x, t)), \quad (18)$$

where $\varepsilon_d > 0$ and $\hat{\mu}_d(t)$ is the estimate of $\mu_d(t)$.

The adaptation laws are given by

$$\begin{aligned} \dot{\hat{\theta}}_1 = -\delta_1 \hat{\theta}_1 - \sigma_1(\gamma_1 q_3; e + v, e + e_t) + \gamma_1 g_1, \\ \hat{\theta}_1(0) = \hat{\theta}_{10}, \end{aligned} \quad (19.a)$$

$$\begin{aligned} \dot{\hat{\theta}}_2 &= -\delta_2 \hat{\theta}_2 - \sigma_2(\gamma_2 q_3; e_t + v_t, e + e_t) + \gamma_2 g_2, \\ \hat{\theta}_2(0) &= \hat{\theta}_{20}, \end{aligned} \tag{19.b}$$

$$\begin{aligned} \dot{\hat{\theta}}_3 &= -\delta_3 \hat{\theta}_3 - \langle \gamma_3 q_3 g, e + e_t \rangle + \gamma_3 g_3, \quad \hat{\theta}_3(0) = \hat{\theta}_{30}, \end{aligned} \tag{19.c}$$

$$\begin{aligned} \dot{\hat{\mu}}_d &= -2\delta_d \hat{\mu}_d + \gamma_d \|e + e_t\| + \gamma_d g_d, \quad \hat{\mu}_d(0) = \hat{\mu}_{d0}, \end{aligned} \tag{19.d}$$

where for $i = 1, 2, 3$, $\delta_i, \gamma_i > 0$

$$\begin{cases} g_i = -\frac{\xi_i^2}{\|\hat{\theta}_i\|_Q \xi_i + \varepsilon_i} \hat{\theta}_i, & \varepsilon_i > 0, \quad \xi_i \geq \|f_i\|_Q, \\ f_i \triangleq -\frac{\delta_i}{\gamma_i} \left(\theta_i + \frac{\dot{\theta}_i}{\delta_i} \right), \end{cases} \tag{20}$$

and $\delta_d > 0, \gamma_d > 0$,

$$\begin{cases} g_d = -\frac{\xi_d^2}{|\hat{\mu}_d| \xi_d + \varepsilon_d} \hat{\mu}_d, & \varepsilon_d > 0, \quad \xi_d \geq |f_d|, \\ f_d \triangleq -\frac{\delta_d}{\gamma_d} \left(\mu_d + \frac{\dot{\mu}_d}{\delta_d} \right). \end{cases} \tag{21}$$

The terms $-\delta_i \hat{\theta}_i$ and $-\delta_d \hat{\mu}_d$ in (19.a–d) are purposely inserted to enhance the convergence of θ_i and μ_d , respectively; g_i and g_d are introduced to cope with the variations of θ_i and μ_d , respectively. Since $\theta_i, \dot{\theta}_i, \mu_d$, and $\dot{\mu}_d$ are assumed to be bounded, ξ_i and ξ_d can be selected at reasonable values by making γ_i and γ_d sufficiently large. It is also noted that the control magnitudes ξ_i and ξ_d are to compensate the maximum possible bounds of f_i and f_d , respectively, for both positive and negative cases.

Although adaptation laws (19.a–d) contain the unknown parameter q_3 , this is not a problem at all because $\gamma_i q_3, i = 1, 2, 3$, are treated as adaptation gains. Thus, the adaptation laws (19.a–d) can be rewritten as

$$\begin{aligned} \dot{\hat{\theta}}_1 &= -\delta_1 \hat{\theta}_1 - \sigma_1(\gamma_{01}; e + v, e + e_t) + \gamma_{01} g_1, \quad \hat{\theta}_1(0) = \hat{\theta}_{10}, \end{aligned} \tag{22.a}$$

$$\begin{aligned} \dot{\hat{\theta}}_2 &= -\delta_2 \hat{\theta}_2 - \sigma_2(\gamma_{02}; e_t + v_t, e + e_t) + \gamma_{02} g_2, \quad \hat{\theta}_2(0) = \hat{\theta}_{20}, \end{aligned} \tag{22.b}$$

$$\begin{aligned} \dot{\hat{\theta}}_3 &= -\delta_3 \hat{\theta}_3 - \langle \gamma_{03} g, e + e_t \rangle + \gamma_{03} g_3, \quad \hat{\theta}_3(0) = \hat{\theta}_{30}, \end{aligned} \tag{22.c}$$

$$\begin{aligned} \dot{\hat{\mu}}_d &= -2\delta_d \hat{\mu}_d + \gamma_d \|e + e_t\| + \gamma_d g_d, \quad \hat{\mu}_d(0) = \hat{\mu}_{d0}, \end{aligned} \tag{22.d}$$

where $\gamma_{0i} = \gamma_i q_3 > 0, i = 1, 2, 3$.

The substitution of (17) into (12) yields the closed loop plant equation as

$$\begin{aligned} \langle u_{tt}, \varphi \rangle &- \sigma_1(q_3 \tilde{\theta}_1; u, \varphi) + \sigma_1(q_1^*; u, \varphi) \\ &- \sigma_2(q_3 \tilde{\theta}_2; u_t, \varphi) + \sigma_2(q_2^*; u_t, \varphi) \\ &= \langle q_3 \tilde{\theta}_3 g, \varphi \rangle + \langle q_3^* g, \varphi \rangle - \langle \mu_0 e_t, \varphi \rangle - \langle \mu e, \varphi \rangle \\ &+ \langle f_c, \varphi \rangle + \langle d, \varphi \rangle, \end{aligned} \tag{23}$$

where $\tilde{\theta}_i(t) = \hat{\theta}_i(t) - \theta_i^*, i = 1, 2, 3$, are the controller parameter estimation errors. In deriving (23), $q_i - q_3 \hat{\theta}_i =$

$q_3 q_3^{-1}(q_i - q_i^*) + q_i^* - q_3 \hat{\theta}_i = q_i^* - q_3(\hat{\theta}_i - \theta_i^*) = q_i^* - q_3 \tilde{\theta}_i, i = 1, 2$, and $q_3 \hat{\theta}_3 = q_3 \tilde{\theta}_3 + q_3 \theta_3^*$ have been used.

From (14), the following state error equation is obtained from (13) and (23).

$$\begin{aligned} \langle e_{tt}, \varphi \rangle &- \sigma_1(q_3 \tilde{\theta}_1; e, \varphi) - \sigma_1(q_3 \tilde{\theta}_1; v, \varphi) + \sigma_1(q_1^*; e, \varphi) \\ &- \sigma_2(q_3 \tilde{\theta}_2; e_t, \varphi) - \sigma_2(q_3 \tilde{\theta}_2; v_t, \varphi) + \sigma_2(q_2^*; e_t, \varphi) \\ &= \langle q_3 \tilde{\theta}_3 g, \varphi \rangle - \langle \mu_0 e_t, \varphi \rangle - \langle \mu e, \varphi \rangle + \langle f_c, \varphi \rangle + \langle d, \varphi \rangle, \\ e(t, 0) &= \frac{\partial e(t, 0)}{\partial x} = 0, \\ e(0, x) &= e_0(x), \quad e_t(0, x) = e_{t0}(x), \end{aligned} \tag{24}$$

where $e_0(x) \in V$ and $e_{t0}(x) \in H$. Using (24), it will be shown that the closed loop system (23), i.e., the plant (12) with the control law (17) and the adaptation laws (22.a–d), will track the state of the reference model (13) in the sense that (15) is satisfied. The well-posedness of the coupled nonlinear system (23) and (22.a–d) is firstly stated.

3. Well-Posedness of Coupled Nonlinear System

For $q \in Q$ and $\psi, \varphi \in V$, let $A_i(q) \in L(V, V^*), i = 1, 2$, be differential operators such that

$$A_i(q) \triangleq \frac{\partial^2}{\partial x^2} \left(q \frac{\partial^2}{\partial x^2} \right), \tag{25.a}$$

$$\langle A_i(q) \psi, \varphi \rangle_{V^*, V} = \sigma_i(q; \psi, \varphi), \tag{25.b}$$

and $D(A_i) = \{ \psi \in V : A_i \psi \in H \}$, where $L(V, V^*)$ denotes the set of continuous linear functions from V to V^* . The existence of $A_i, i = 1, 2$, is guaranteed by the boundedness of $\sigma_i, i = 1, 2$. Using the operators $A_i(q), i = 1, 2$, the state error equation (24) can be rewritten as

$$\langle e_{tt}, \varphi \rangle = -\langle A_1(q_1^*) e, \varphi \rangle - \langle A_2(q_2^*) e_t, \varphi \rangle + \langle f_0(t, e, \hat{\theta}_i), \varphi \rangle, \tag{26}$$

where $f_0(t, e, \hat{\theta}_i) = A_1(q_3(\hat{\theta}_1 - \theta_1^*))e + A_2(q_3(\hat{\theta}_2 - \theta_2^*))e_t + q_3(\hat{\theta}_3 - \theta_3^*)g + A_1(q_3(\hat{\theta}_1 - \theta_1^*))v + A_2(q_3(\hat{\theta}_2 - \theta_2^*))v_t - \mu_0 e_t - \mu e + f_c + d$.

Let $\{X, \langle \cdot, \cdot \rangle_X\}$ be the Hilbert space defined by $X = V \times H$ and

$$\langle \psi_0, \varphi_0 \rangle_X = \langle \psi_1, \varphi_1 \rangle_V + \langle \psi_2, \varphi_2 \rangle_H \tag{27}$$

for $\psi_0 = (\psi_1, \psi_2) \in X$ and $\varphi_0 = (\varphi_1, \varphi_2) \in X$. And, let $\{Y, \langle \cdot, \cdot \rangle_Y\}$ be the Hilbert space defined by $Y = V \times V$ and

$$\langle \psi_0, \varphi_0 \rangle_Y = \langle \psi_1, \varphi_1 \rangle_V + \langle \psi_2, \varphi_2 \rangle_V \tag{28}$$

for $\psi_0 = (\psi_1, \psi_2) \in Y$ and $\varphi_0 = (\varphi_1, \varphi_2) \in Y$, and $Y \hookrightarrow X \hookrightarrow Y^*$ with the embedding dense and continuous. The following is then satisfied:

$$\|\psi_0\|_X \leq K_0 \|\psi_0\|_Y \tag{29}$$

for $\psi_0 \in Y$ and $K_0 > 0$.

From (26), the state error system can be rewritten in the first order form as

$$\langle e_t, \varphi_0 \rangle_{Y^*, Y} = \langle A_0 e, \varphi_0 \rangle_{Y^*, Y} + \langle f, \varphi_0 \rangle_{Y^*, Y}, \quad e(0) = e_0, \tag{30}$$

where $e = (e, e_t)^T$, $\varphi_0 = (\varphi_1, \varphi_2)^T$, $f = (0, f_0)^T$, and

$$A_0 = \begin{bmatrix} 0 & 1 \\ -A_1(q_1^*) & -A_2(q_2^*) \end{bmatrix},$$

where $A_0 \in L(Y, Y^*)$ and $D(A_0) = \{\psi_0 = (\psi_1, \psi_2) \in Y : A_1(q_1^*)\psi_1 + A_2(q_2^*)\psi_2 \in H\}$. The nonlinear coupled system (30) and (22.a-d) can be then given as follows:

$$\langle e_t, \varphi_0 \rangle = \langle A_0 e, \varphi_0 \rangle + \langle f, \varphi_0 \rangle, \tag{31.a}$$

$$\langle \dot{\hat{\theta}}_1, p \rangle_Q = -\langle \delta_1 \hat{\theta}_1, p \rangle_Q - \langle \gamma_{01}(e_{xx} + v_{xx})(e_{xx} + e_{txx}), p \rangle_Q + \langle \gamma_{01}g_1, p \rangle_Q, \tag{31.b}$$

$$\langle \dot{\hat{\theta}}_2, p \rangle_Q = -\langle \delta_2 \hat{\theta}_2, p \rangle_Q - \langle \gamma_{02}(e_{txx} + v_{txx})(e_{xx} + e_{txx}), p \rangle_Q + \langle \gamma_{02}g_2, p \rangle_Q, \tag{31.c}$$

$$\langle \dot{\hat{\theta}}_3, p \rangle_Q = -\langle \delta_3 \hat{\theta}_3, p \rangle_Q - \langle \gamma_{03}g(e + e_t), p \rangle_Q + \langle \gamma_{03}g_3, p \rangle_Q, \tag{31.d}$$

$$\dot{\hat{\mu}}_d r_d = -2\delta_d \hat{\mu}_d r_d + \gamma_d \|e + e_t\| r_d + \gamma_d g_d r_d, \tag{31.e}$$

$$e(0) = e_0, \quad \hat{\theta}_1(0) = \hat{\theta}_{10}, \quad \hat{\theta}_2(0) = \hat{\theta}_{20}, \tag{31.f}$$

$$\hat{\theta}_3(0) = \hat{\theta}_{30}, \quad \hat{\mu}_d(0) = \hat{\mu}_{d0},$$

where $e_{xx} = \frac{\partial^2 e}{\partial x^2}$, $e_{txx} = \frac{\partial^3 e}{\partial x^2 \partial t}$, and $r_d \in R$. From the system (31.f), the following is obtained:

$$\begin{aligned} &\langle e_t, \varphi_0 \rangle + \langle \dot{\hat{\theta}}_1, p \rangle_Q + \langle \dot{\hat{\theta}}_2, p \rangle_Q + \langle \dot{\hat{\theta}}_3, p \rangle_Q + \dot{\hat{\mu}}_d r_d \\ &= \langle A_0 e, \varphi_0 \rangle - \langle \delta_1 \hat{\theta}_1, p \rangle_Q - \langle \delta_2 \hat{\theta}_2, p \rangle_Q \\ &\quad - \langle \delta_3 \hat{\theta}_3, p \rangle_Q - 2\delta_d \hat{\mu}_d r_d \\ &\quad + \langle f, \varphi_0 \rangle - \langle \gamma_{01}(e_{xx} + v_{xx})(e_{xx} + e_{txx}), p \rangle_Q + \langle \gamma_{01}g_1, p \rangle_Q \\ &\quad - \langle \gamma_{02}(e_{txx} + v_{txx})(e_{xx} + e_{txx}), p \rangle_Q + \langle \gamma_{02}g_2, p \rangle_Q \\ &\quad - \langle \gamma_{03}g(e + e_t), p \rangle_Q + \langle \gamma_{03}g_3, p \rangle_Q + \gamma_d \|e + e_t\| r_d + \gamma_d g_d r_d. \end{aligned} \tag{32}$$

Define a state space as $W \triangleq Y \times Q^3 \times R$. The system (32) can be then rewritten as

$$\begin{aligned} &\left(\begin{array}{c} \dot{e} \\ \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \\ \dot{\hat{\theta}}_3 \\ \dot{\hat{\mu}}_d \end{array} \right) \begin{array}{c} \left(\begin{array}{c} \varphi_0 \\ p \\ p \\ p \\ r_d \end{array} \right) \end{array} \Bigg|_{W^*, W} \\ &= \left(\begin{array}{ccccc} A_0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_1 & 0 & 0 & 0 \\ 0 & 0 & -\delta_2 & 0 & 0 \\ 0 & 0 & 0 & -\delta_3 & 0 \\ 0 & 0 & 0 & 0 & -2\delta_d \end{array} \right) \begin{array}{c} \left(\begin{array}{c} \dot{e} \\ \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \\ \dot{\hat{\theta}}_3 \\ \dot{\hat{\mu}}_d \end{array} \right) \left(\begin{array}{c} \varphi_0 \\ p \\ p \\ p \\ r_d \end{array} \right) \end{array} \Bigg|_{W^*, W} \\ &+ \left(\begin{array}{c} f \\ -\gamma_{01}(e_{xx} + v_{xx})(e_{xx} + e_{txx}) + \gamma_{01}g_1 \\ -\gamma_{02}(e_{txx} + v_{txx})(e_{xx} + e_{txx}) + \gamma_{02}g_2 \\ -\gamma_{03}g(e + e_t) + \gamma_{03}g_3 \\ \gamma_d \|e + e_t\| + \gamma_d g_d \end{array} \right) \begin{array}{c} \left(\begin{array}{c} \varphi_0 \\ p \\ p \\ p \\ r_d \end{array} \right) \end{array} \Bigg|_{W^*, W} \end{aligned} \tag{33}$$

where $(\varphi_0, p, p, p, r_d)^T \in W$.

The weak form (33) is formally equivalent to the system

$$\dot{z} = Az + F(t, z), \quad z(0) = z_0, \tag{34}$$

where $z = (e, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\mu}_d)^T \in Z$, $Z \triangleq X \times Q^3 \times R$, and

$$A = \begin{bmatrix} A_0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_1 & 0 & 0 & 0 \\ 0 & 0 & -\delta_2 & 0 & 0 \\ 0 & 0 & 0 & -\delta_3 & 0 \\ 0 & 0 & 0 & 0 & -2\delta_d \end{bmatrix},$$

$$F(t, z) = \begin{bmatrix} f \\ -\gamma_{01}(e_{xx} + v_{xx})(e_{xx} + e_{txx}) + \gamma_{01}g_1 \\ -\gamma_{02}(e_{txx} + v_{txx})(e_{xx} + e_{txx}) + \gamma_{02}g_2 \\ -\gamma_{03}g(e + e_t) + \gamma_{03}g_3 \\ \gamma_d \|e + e_t\| + \gamma_d g_d \end{bmatrix},$$

where $A: D(A) \subset Z \rightarrow Z$ and

$$D(A) = \{(\psi_0, \theta) \in Z : \psi_0 \in (\psi_1, \psi_2) \in X \text{ and } A_1(q_1^*)\psi_1 + A_2(q_2^*)\psi_2 \in H\}.$$

$D(A)$ is dense, and A is a closed operator⁽¹⁸⁾. Therefore, the existence of a unique solution to the system (33) can be established by establishing the existence of a unique strong solution to the initial value problem in Z given by (34).

For $z \in D(A)$

$$\begin{aligned} \langle z, Az \rangle_Z &= \langle e_t, e \rangle_V - \langle A_1(q_1^*)e, e_t \rangle - \langle A_2(q_2^*)e_t, e_t \rangle \\ &\quad - \sum_{i=1}^3 \delta_i \|\hat{\theta}_i\|_Q^2 - 2\delta_d \hat{\mu}_d^2 \\ &= -\sigma_2(q_2^*; e_t, e_t) - \sum_{i=1}^3 \delta_i \|\hat{\theta}_i\|_Q^2 - 2\delta_d \hat{\mu}_d^2 \\ &\leq -k_2(q_2^*) \|e_t\|_V^2 - \sum_{i=1}^3 \delta_i \|\hat{\theta}_i\|_Q^2 - 2\delta_d \hat{\mu}_d^2. \end{aligned} \tag{35}$$

Similarly, it can be shown that the dual of A_0 given by

$$A_0^* = \begin{bmatrix} 0 & -1 \\ A_1(q_1^*) & -A_2(q_2^*) \end{bmatrix} \tag{36}$$

satisfies $\langle e, A_0^* e \rangle_X \leq -k_2(q_2^*) \|e_t\|_V^2$ and so $A_0(t)$ generates a contraction semigroup.

Hence, $A: D(A) \subset Z \rightarrow Z$ is the infinitesimal generator of a linear process $\{S(t)\}_{t \geq 0} = \{(\Phi(t, 0), \Theta_1(t), \Theta_2(t), \Theta_3(t), E(t))\}_{t \geq 0}$ on Z ⁽¹⁸⁾. Note that the first component $\Phi(t, 0)$ is generated by A_0 . Note also that $\Phi(t, 0)e_0$ is the strong solution of the evolution equation $\dot{e}(t) = A_0 e(t)$ for every $e_0 \in D(A_0)$.

Now, set $z = (e, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\mu}_d)^T$ and $z' = (e', \hat{\theta}'_1, \hat{\theta}'_2, \hat{\theta}'_3, \hat{\mu}'_d)^T$. Then,

$$\begin{aligned} &\|F(t, z) - F(t, z')\|_Z^2 \\ &\leq [\beta_e^2 (\|\hat{\theta}'_1\|_Q^2 + \|\hat{\theta}'_2\|_Q^2 + K^2 \mu^2) \\ &\quad + \|\gamma_{01}\|_Q^2 (\|e_{xx}\|^2 + \|e'_{xx}\|^2 + \|e_{txx}\|^2 + \|v_{xx}\|^2) \\ &\quad + \|\gamma_{02}\|_Q^2 (\|v_{txx}\|^2 + \|e'_{txx}\|^2) + K^2 \|\gamma_{03}\|_Q^2 \|g\|^2] \|e - e'\|_V^2 \end{aligned}$$

$$\begin{aligned}
 &+ \left[\left(K_a^2 \beta_e^2 (\|\hat{\theta}_2\|_Q^2 + \|q_2^*\|^2) + \mu_0^2 \right) \right. \\
 &+ K_a^2 \|\gamma_{01}\|_Q^2 (\|v_{xx}\|^2 + \|e'_{xx}\|^2) \\
 &+ K_a^2 \|\gamma_{02}\|_Q^2 (\|e_{xx}\|^2 + \|e'_{txx}\|^2 + \|e_{txx}\|^2 + \|v_{txx}\|^2) \\
 &+ \|\gamma_{03}\|_Q^2 \|g\|^2 \left. \right] \|e_t - e'_t\|^2 \\
 &+ \left[2 \left| \frac{\hat{\mu}'_d}{(\hat{\mu}'_d \|e + e_t + \varepsilon_d)(\hat{\mu}'_d \|e' + e'_t + \varepsilon_d)} \right|^2 \right. \\
 &+ \gamma_d^2 K_d \\
 &+ \left. \left| \frac{\varepsilon_d}{(\hat{\mu}'_d \|e + e_t + \varepsilon_d)(\hat{\mu}'_d \|e' + e'_t + \varepsilon_d)} \right|^2 \right. \\
 &\times (K^2 \|e - e'\|_V^2 + \|e_t - e'_t\|^2) \\
 &+ \beta_a^2 (\|e_{xx}\|^2 + \|v_{xx}\|^2) \|\hat{\theta}_1 - \hat{\theta}'_1\|_Q^2 \\
 &+ \beta_a^2 (\|e_{txx}\|^2 + \|v_{txx}\|^2) \|\hat{\theta}_2 - \hat{\theta}'_2\|_Q^2 \\
 &+ \|q_3\|_Q^2 \|g\|^2 \|\hat{\theta}_3 - \hat{\theta}'_3\|_Q^2 \\
 &+ \sum_{i=1}^3 \|\gamma_{0i}\|_Q^2 \left(\frac{2\xi_i^2 \|\hat{\theta}_i\|_Q^2 + \varepsilon_i^2 \xi_i^4}{(\|\hat{\theta}_i\|_Q \xi_i + \varepsilon_i)(\|\hat{\theta}'_i\|_Q \xi_i + \varepsilon_i)} \right) \|\hat{\theta}_i - \hat{\theta}'_i\|_Q^2 \\
 &+ \left[\left| \frac{\hat{\mu}_d \hat{\mu}'_d \|e + e_t + \varepsilon_d(\hat{\mu}_d + \hat{\mu}'_d)}{(\hat{\mu}_d \|e + e_t + \varepsilon_d)(\hat{\mu}'_d \|e + e_t + \varepsilon_d)} \right|^2 \right. \\
 &+ \gamma_d^2 \left(\frac{2\xi_d^6 \|\hat{\mu}'_d\|^2 + \varepsilon_d^2 \xi_d^4}{(\hat{\mu}_d \xi_d + \varepsilon_d)(\hat{\mu}'_d \xi_d + \varepsilon_d)} \right) \left. \right] \|\hat{\mu}_d - \hat{\mu}'_d\|^2. \tag{37}
 \end{aligned}$$

Hence,

$$\|F(t, z) - F(t, z')\|_Z \leq C_1 \|z - z'\|_Z, \tag{38}$$

where C_1 is a positive constant. Therefore $F: Z \rightarrow Z$ is locally Lipschitz continuous in Z . Thus a unique solution exists. Finally, the strong solution of (30) can be written in the following variation of constant formula⁽¹⁹⁾

$$e(t) = \Phi(t, 0)e(0) + \int_0^t \Phi(t, \tau) f(\tau, e(\tau), \hat{\theta}_i(\tau)) d\tau, \tag{39}$$

where $\Phi(t, s)$ is the evolution operator associated with A_0 in the space X .

4. Tracking and Parameter Errors Convergence

The closed loop boundedness of u , u_t , $\hat{\theta}_i$'s, $\hat{\mu}_d$, and f , and the convergence of e and $\tilde{\theta}_i$'s are now considered. By considering an appropriate Lyapunov function, the stability of the closed loop system can be established. A functional $V: [0, \infty) \rightarrow R^+$ is now considered as

$$V(t) = \frac{1}{2} V_1(t) + \sum_{i=1}^3 \frac{1}{2\gamma_i} \langle \tilde{\theta}_i, \tilde{\theta}_i \rangle_Q + \frac{1}{2\gamma_d} \tilde{\mu}_d^2 \tag{40}$$

for $i = 1, 2, 3$, and where

$$\begin{aligned}
 V_1(t) = &\langle (\mu_0 + \mu - 1)e, e \rangle + \langle e_t + e, e_t + e \rangle \\
 &+ \sigma_1 \langle q_1^*; e, e \rangle + \sigma_2 \langle q_2^*; e, e \rangle.
 \end{aligned}$$

Differentiating (40) with respect to t along the trajectories of (24) yields:

$$\begin{aligned}
 \dot{V}(t) = &\langle (\mu_0 + \mu - 1)e, e_t \rangle + \langle e_{tt}, e_t \rangle + \langle e_t, e_t \rangle + \langle e_{tt}, e \rangle + \langle e, e_t \rangle \\
 &+ \sigma_1 \langle q_1^*; e, e_t \rangle + \sigma_2 \langle q_2^*; e, e_t \rangle + \sum_{i=1}^3 \frac{1}{\gamma_i} \langle \dot{\tilde{\theta}}_i, \tilde{\theta}_i \rangle_Q + \frac{1}{\gamma_d} \dot{\tilde{\mu}}_d \tilde{\mu}_d \\
 = &-\langle \mu e, e \rangle - \langle (\mu_0 - 1)e_t, e_t \rangle - \sigma_1 \langle q_1^*; e, e \rangle - \sigma_2 \langle q_2^*; e_t, e_t \rangle \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{\gamma_1} \langle \dot{\tilde{\theta}}_1, \tilde{\theta}_1 \rangle_Q - \frac{1}{\gamma_1} \langle \dot{\theta}_1, \tilde{\theta}_1 \rangle_Q + \sigma_1 \langle q_3 \tilde{\theta}_1; e + v, e + e_t \rangle \\
 &+ \frac{1}{\gamma_2} \langle \dot{\tilde{\theta}}_2, \tilde{\theta}_2 \rangle_Q - \frac{1}{\gamma_2} \langle \dot{\theta}_2, \tilde{\theta}_2 \rangle_Q + \sigma_2 \langle q_3 \tilde{\theta}_2; e_t + v_t, e + e_t \rangle \\
 &+ \frac{1}{\gamma_3} \langle \dot{\tilde{\theta}}_3, \tilde{\theta}_3 \rangle_Q - \frac{1}{\gamma_3} \langle \dot{\theta}_3, \tilde{\theta}_3 \rangle_Q + \langle q_3 \tilde{\theta}_3 g, e + e_t \rangle \\
 &+ \langle f_c, e + e_t \rangle + \langle d, e + e_t \rangle + \frac{1}{\gamma_d} \tilde{\mu}_d (\dot{\hat{\mu}}_d - \dot{\mu}_d). \tag{42}
 \end{aligned}$$

Using condition (A2) and (10), (41) becomes

$$\begin{aligned}
 &-\langle \mu e, e \rangle - \langle (\mu_0 - 1)e_t, e_t \rangle - \sigma_1 \langle q_1^*; e, e \rangle - \sigma_2 \langle q_2^*; e_t, e_t \rangle \\
 &\leq -\mu \|e\|^2 - \left(\mu_0 - 1 + \frac{k_2(q_2^*)}{K^2} \right) \|e_t\|^2 - k_2(q_1^*) \|e_t\|_V^2. \tag{43}
 \end{aligned}$$

Therefore, using (43) and adaptation laws (22.a-d), (42) yields:

$$\begin{aligned}
 \dot{V} \leq &-\mu \|e\|^2 - \left(\mu_0 - 1 + \frac{k_2(q_2^*)}{K^2} \right) \|e_t\|^2 - k_2(q_1^*) \|e\|_V^2 \\
 &+ \langle f_c, e + e_t \rangle + \langle d, e + e_t \rangle + \frac{1}{\gamma_d} \tilde{\mu}_d (-\delta_d \hat{\mu}_d + \gamma_d \|e + e_t\|) \\
 &- \frac{\delta_d}{\gamma_d} \hat{\mu}_d \tilde{\mu}_d + g_d \tilde{\mu}_d - \frac{1}{\gamma_d} \dot{\mu}_d \tilde{\mu}_d \\
 &+ \sum_{i=1}^3 \left(-\frac{\delta_i}{\gamma_i} \langle \hat{\theta}_i, \tilde{\theta}_i \rangle_Q + \langle g_i, \tilde{\theta}_i \rangle_Q - \frac{1}{\gamma_a} \langle \dot{\theta}_i, \tilde{\theta}_i \rangle_Q \right). \tag{44}
 \end{aligned}$$

The right hand side terms of (44) satisfy the following inequalities, respectively:

$$\begin{aligned}
 &\langle f_c, e + e_t \rangle + \langle d, e + e_t \rangle + \frac{1}{\gamma_d} \tilde{\mu}_d (-\delta_d \hat{\mu}_d + \gamma_d \|e + e_t\|) \\
 &\leq -\frac{\delta_d}{\gamma_d} \tilde{\mu}_d^2 + \varepsilon_d + \frac{\delta_d}{2\gamma_d} \mu_d, \tag{45.a}
 \end{aligned}$$

$$\begin{aligned}
 &-\frac{\delta_i}{\gamma_i} \langle \hat{\theta}_i, \tilde{\theta}_i \rangle_Q + \langle g_i, \tilde{\theta}_i \rangle_Q - \frac{1}{\gamma_a} \langle \dot{\theta}_i, \tilde{\theta}_i \rangle_Q \\
 &\leq -\frac{\delta_i}{\gamma_i} \|\tilde{\theta}_i\|_Q^2 + \varepsilon_i + \xi_i \|\theta_i\|_Q + \frac{\delta_i}{\gamma_i} \|\theta_i\|_Q^2 + \frac{1}{\gamma_i} \langle \dot{\theta}_i, \theta_i \rangle_Q, \tag{45.b}
 \end{aligned}$$

$$\begin{aligned}
 &-\frac{\delta_d}{\gamma_d} \hat{\mu}_d \tilde{\mu}_d + g_d \tilde{\mu}_d - \frac{1}{\gamma_d} \dot{\mu}_d \tilde{\mu}_d \\
 &\leq -\frac{\delta_d}{\gamma_d} \tilde{\mu}_d^2 + \varepsilon_d + \xi_d |\mu_d| + \frac{\delta_d}{\gamma_d} \mu_d^2 + \frac{1}{\gamma_d} \dot{\mu}_d \mu_d. \tag{45.c}
 \end{aligned}$$

Therefore, the derivative of the Lyapunov function candidate is bounded as follows:

$$\begin{aligned}
 \dot{V} \leq &-\mu \|e\|^2 - \left(\mu_0 - 1 + \frac{k_2(q_2^*)}{K^2} \right) \|e_t\|^2 - k_2(q_1^*) \|e\|_V^2 \\
 &- \frac{2\delta_d}{\gamma_d} \tilde{\mu}_d^2 - \sum_{i=1}^3 \frac{\delta_i}{\gamma_i} \|\tilde{\theta}_i\|_Q^2 + v(t), \tag{46}
 \end{aligned}$$

where

$$v(t) = 2\varepsilon_d + \frac{\delta_d}{2\gamma_d}\mu_d + \xi_d|\mu_d| + \frac{\delta_d}{\gamma_d}\mu_d^2 + \frac{1}{\gamma_d}\dot{\mu}_d\mu_d + \sum_{i=1}^3 \left(\varepsilon_i + \xi_i\|\theta_i\|_Q + \frac{\delta_i}{\gamma_i}\|\theta_i\|_Q^2 + \frac{1}{\gamma_i}\langle \dot{\theta}_i, \theta_i \rangle_Q \right). \quad (47)$$

Note that $v(t)$ is bounded because of the assumption that $\theta_i, \dot{\theta}_i, \mu_d,$ and $\dot{\mu}_d$ are bounded.

Remark 1: Since $v(t)$ is bounded, the solutions for coupled nonautonomous dynamical systems (24) and (22.a–d) are uniformly ultimately bounded. Further, if $v(t)$ is sufficiently small, then it is guaranteed that $\|e\|_Y$ and $\|e_t\|$ are uniformly ultimately bounded within an arbitrarily small neighborhood of zero^{(7),(8)}.

Remark 2: The equations of $\|f_i\|_Q$'s and $\|f_d\|_Q$ can be rewritten as

$$\|f_i\|_Q = \left\| -\frac{\delta_i}{\gamma_i} \left(\theta_i + \frac{\dot{\theta}_i}{\delta_i} \right) \right\|_Q = \frac{1}{\gamma_i} \left| -\delta_i \theta_i + \dot{\theta}_i \right|_Q \quad \text{for } i = 1, 2, 3,$$

$$\text{and } |f_d| = \left| -\frac{\delta_d}{\gamma_d} \left(\mu_d + \frac{\dot{\mu}_d}{\delta_d} \right) \right| = \frac{1}{\gamma_d} | -\delta_d \mu_d + \dot{\mu}_d |.$$

From $\xi_i \geq \|f_i\|_Q, i = 1, 2, 3,$ and $\xi_d \geq |f_d|,$ ξ_i 's and ξ_d can be chosen at reasonable values according to $\|f_i\|_Q$'s and $|f_d|,$ respectively. Thus, $v(t)$ can be pushed in an arbitrarily small boundedness region by making sufficiently small ε_i 's, $\varepsilon_d,$ δ_i 's, δ_d and sufficiently large γ_i 's, $\gamma_d.$

All the above developments are now summarized as follows:

Theorem 1: Consider the nonlinear coupled dynamical system (24), (13), and (22.a–d) (or (19.a–c)). Then all signals in the closed loop system are bounded. Furthermore, both the state error and its time-derivative, $(e, e_t),$ converge asymptotically near to zero by a suitable choice of $\varepsilon_i, \delta_i, \gamma_i, i = 1, 2, 3,$ and $\varepsilon_d, \delta_d, \gamma_d,$ i.e.,

$$\lim_{t \rightarrow \infty} \|e_t\| = \lim_{t \rightarrow \infty} \|e\|_Y \approx 0.$$

Theorem 1 implies that the basic control objective is now achieved, i.e., all the signals in the closed loop are bounded and the trajectory following is achieved. In addition to the state error convergence near to zero, it is also desirable to have an adaptive control scheme to provide parameter estimation error convergence near to zero as well, i.e., the parameters $\hat{\theta}_i, i = 1, 2, 3,$ and $\hat{\mu}_d$ converge near to the true parameters $\theta_i^*, i = 1, 2, 3,$ and $\mu_d,$ respectively. If the parameter errors convergence is established, the entire adaptive algorithm can be improved. To assure this, the following additional persistency of excitation condition on the reference model is required.

Using the operator $A_i(\cdot), i = 1, 2,$ given in (25.a, b), let $F_0(\cdot), F_1(\cdot) \in L^2(0, T; Y^*)$ for all $T > 0$ be given by

$$F_0(p_0) = \begin{bmatrix} 0 \\ A(p_1)\{e+v\} + A(p_2)\{e_t+v_t\} + p_3g \end{bmatrix}, \quad (48)$$

$$F_1(p_0) = \begin{bmatrix} A(p_1)\{e+v\} + A(p_2)\{e_t+v_t\} + p_3g \\ 0 \end{bmatrix}, \quad (49)$$

for $p_0 = (p_1, p_2, p_3, r_d)^T \in \mathcal{Q}, \mathcal{Q} \triangleq \mathcal{Q}^3 \times R,$ and let a matrices B and D be given by

$$B = \begin{bmatrix} 0 & 0 \\ -\mu & -\mu_0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ f_c + d \end{bmatrix}. \quad (50)$$

Then, the coupled system (24), (13), and (22.a–d) can be rewritten as

$$\langle e_t, \varphi_0 \rangle = \langle A_0(t)e, \varphi_0 \rangle + \langle F_0(q_3\tilde{\theta}), \varphi_0 \rangle + \langle Be, \varphi_0 \rangle + \langle D, \varphi_0 \rangle, \quad (51.a)$$

$$\langle v_t, \varphi_0 \rangle = \langle A_0(t)v, \varphi_0 \rangle + \langle g_0, \varphi_0 \rangle, \quad (51.b)$$

$$\langle \dot{\tilde{\theta}}, p_0 \rangle_Q = \langle F_0(\gamma_0 p_0), e \rangle + \langle F_1(\gamma_0 p_0), e \rangle + \langle -\delta\hat{\theta} + \gamma_0\|e + e_t\| + \gamma g - \hat{\theta}, q \rangle_Q, \quad (51.c)$$

$$e(0) = e_0, \quad v(0) = v_0, \quad \tilde{\theta}(0) = \tilde{\theta}_0, \quad (51.d)$$

where $\varphi_0 \in Y, p_0 = (p_1, p_2, p_3, r_d)^T \in \mathcal{Q}, \tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\mu}_d)^T \in \mathcal{Q}, v = (v, v_t)^T, g_0 = (0, q_3^*g)^T, \gamma_0 p_0 = (\gamma_{01}p_1, \gamma_{02}p_2, \gamma_{03}p_3)^T, \delta\hat{\theta} = (\delta_1\hat{\theta}_1, \delta_2\hat{\theta}_2, \delta_3\hat{\theta}_3, \delta_d\hat{\mu}_d), \gamma_0 = (0, 0, 0, \gamma_d), \gamma g = (\gamma_{01}g_1, \gamma_{02}g_2, \gamma_{03}g_3, 0),$ and $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\mu}_d).$

From (10) and (A1), the followings are obtained:

$$|\langle A_0\psi_0, \varphi_0 \rangle| \leq k_3\|\psi_0\|_Y\|\varphi_0\|_Y, \quad (52)$$

$$|\langle B\psi_0, \varphi_0 \rangle| \leq k_4\|\psi_0\|_Y\|\varphi_0\|_Y, \quad (53)$$

$$|\langle F_0(p_0), \varphi_0 \rangle| \leq k_5\|p_0\|_Q(\|e\|_Y + \|v\|_Y + \|g_0\|_Y)\|\varphi_0\|_Y, \quad (54)$$

$$|\langle F_1(p_0), \varphi_0 \rangle| \leq k_5\|p_0\|_Q(\|e\|_Y + \|v\|_Y + \|g_0\|_Y)\|\varphi_0\|_Y, \quad (55)$$

for $\psi_0, \varphi_0 \in Y,$ and where $k_i > 0, i = 3, 4, 5.$

Also, note that the following is obtained:

$$|\langle D, \varphi_0 \rangle| \leq \|f_c + d\| \|\varphi_0\|_Y \leq (\hat{\mu}_{d,\max} + \mu_{d,\max})\|\varphi_0\|_Y, \quad (56)$$

where $\|f_c(x, t)\| \leq |\hat{\mu}_d(t)| \leq \hat{\mu}_{d,\max}$ and $\|d(x, t)\| \leq \mu_d(t) \leq \mu_{d,\max}$ for $t \geq 0$ with some positive constants $\hat{\mu}_{d,\max}$ and $\mu_{d,\max}.$

Definition 2: The reference model (51.b), or the triple $\{A_0, g_0, v_0\},$ is said to be persistently exciting if there exist positive constants $\tau_0, \delta_0, c_0,$ and $\varepsilon_0,$ such that for each $p_0 \in \mathcal{Q}$ with $\|p_0\|_Q = 1$ and $t \geq 0$ sufficiently large, there exists $\bar{t} \in [t, t + \tau_0]$ for which

$$\left\| \int_{\bar{t}}^{\bar{t} + \delta_0} F_0(p_0) d\tau \right\|_{Y^*} \geq \varepsilon_0 + (\hat{\mu}_{d,\max} + \mu_{d,\max}) \frac{\delta_0}{c_0}. \quad (57)$$

Theorem 3: If $g_0 \in L_\infty(0, \infty; Y)$ and $v_0 \in Y,$ and if the reference model, (51.b) (or (13)), is persistently exciting, then the uniform ultimate boundedness region of the parameter estimation error vector $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\mu}_d) \in \mathcal{Q}$ can be made arbitrarily small near to zero by a suitable choice of $\varepsilon_i, \delta_i, \gamma_i, i = 1, 2, 3,$ and $\varepsilon_d, \delta_d, \gamma_d,$ then $\lim_{t \rightarrow \infty} \|\tilde{\theta}(0)\|_Q \approx 0.$

Proof: In this proof, it is assumed that $\|\cdot\|_0 \triangleq \|\cdot\|, \|\cdot\|_2 \triangleq \|\cdot\|_X, \|\cdot\| \triangleq \|\cdot\|_Y,$ and $\|\cdot\|_* \triangleq \|\cdot\|_{Y^*}.$ From section 3, the strong solution of (51.b) can be obtained by

$$v(t) = \Phi(t, 0)v(0) + \int_0^t \Phi(t, \tau)g_0(\tau) dt. \quad (58)$$

Now suppose that $g_0 \in L_\infty(0, \infty; Y)$ and $v_0 \in Y$, then $v \in L_\infty(0, \infty; Y)$ follows immediately from (58).

For $t_2 > t_1$, (51.a), (52), (53), (56), and (29) imply that

$$\begin{aligned} & \left\| \int_{t_1}^{t_2} F_0(q_3 \tilde{\theta}(t)) dt \right\|_* \\ & \leq K_0 |e(t_2)|_2 + K_0 |e(t_1)|_2 \\ & + (k_3 + k_4)(t_2 - t_1)^{1/2} \left\{ \int_{t_1}^{t_2} \|e(t)\|^2 dt \right\}^{1/2} \\ & + (\hat{\mu}_{d,\max} + \mu_{d,\max})(t_2 - t_1). \end{aligned} \tag{59}$$

Assume that $\|\theta(t)/\gamma\|_{\mathcal{Q}}$ is uniformly bounded by ρ where $\theta/\gamma = (\tilde{\theta}_1/\gamma_{01}, \tilde{\theta}_2/\gamma_{02}, \tilde{\theta}_3/\gamma_{03}, \tilde{\mu}_d/\gamma_d)^T$. Then, from (51.c), (54), and (55) it follows that for $p_0 = \frac{1}{\rho} \left(\frac{\tilde{\theta}(t)}{\gamma} \right) \in \mathcal{Q}$

$$\begin{aligned} \|\tilde{\theta}(t_2) - \tilde{\theta}(t_1)\|_{\mathcal{Q}} &= \sup_{|p_0|_{\mathcal{Q}} \leq 1} \left| \left\langle \int_{t_1}^{t_2} \tilde{\theta}_i(t) dt, p_0 \right\rangle_{\mathcal{Q}} \right| \\ &\leq \int_{t_1}^{t_2} \sup_{|p_0|_{\mathcal{Q}} \leq 1} |\langle F_0(\gamma_0 p_0), e(t) \rangle| dt \\ &+ \int_{t_1}^{t_2} \sup_{|p_0|_{\mathcal{Q}} \leq 1} |\langle F_1(\gamma_0 p_0), e(t) \rangle| dt \\ &+ \int_{t_1}^{t_2} \sup_{|p_0|_{\mathcal{Q}} \leq 1} | \langle -\delta \hat{\theta} + \gamma_0 |e + e_t|_0 + \gamma g - \dot{\theta}, p_0 \rangle_{\mathcal{Q}} | dt \\ &\leq 2k_5 \int_{t_1}^{t_2} \|e(t)\|^2 dt + 2k_5 (\|v(t)\|_{L_\infty(0,\infty;Y)} \\ &+ \|g_0\|_{L_\infty(0,\infty;Y)})(t_2 - t_1)^{1/2} \left\{ \int_{t_1}^{t_2} \|e(t)\|^2 dt \right\}^{1/2} \\ &+ \frac{1}{\rho} \int_{t_1}^{t_2} | \langle -\delta \hat{\theta} + \gamma_0 |e + e_t|_0 + \gamma g - \dot{\theta}, \tilde{\theta}/\gamma \rangle_{\mathcal{Q}} | dt. \end{aligned} \tag{60}$$

Assume that $\lim_{t \rightarrow \infty} \|\tilde{\theta}(t)\|_{\mathcal{Q}} \neq 0$, and let $\{t_k\}_{k=1}^\infty$ be an increasing sequence of positive numbers for which $\lim_{k \rightarrow \infty} t_k = \infty$ and

$$\|q_3 \tilde{\theta}(t_k)\|_{\mathcal{Q}} \geq c_0, \quad k = 1, 2, \dots$$

Assume further that the reference model (51.b), or (13), is persistently exciting, and for each $k = 1, 2, \dots$, let $\bar{t}_k \in [t_k, t_k + \tau_0]$ be such that

$$\left\| \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} F_0 \left(\frac{q_3 \tilde{\theta}(t_k)}{|q_3 \tilde{\theta}(t_k)|_{\mathcal{Q}}} \right) dt \right\|_* \geq \varepsilon_0 + (\hat{\mu}_{d,\max} + \mu_{d,\max}) \frac{\delta_0}{c_0}. \tag{61}$$

Then, using (59) and (60), the following is derived:

$$\begin{aligned} & 0 < c_0 \varepsilon_0 + (\hat{\mu}_{d,\max} + \mu_{d,\max}) \delta_0 \\ & = c_0 \left(\varepsilon_0 + (\hat{\mu}_{d,\max} + \mu_{d,\max}) \frac{\delta_0}{c_0} \right) \\ & \leq |q_3 \tilde{\theta}(t_k)|_{\mathcal{Q}} \left\| \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} F_0 \left(\frac{q_3 \tilde{\theta}(t_k)}{|q_3 \tilde{\theta}(t_k)|_{\mathcal{Q}}} \right) dt \right\|_* \\ & = \left\| \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} F_0(q_3 \tilde{\theta}(t_k)) dt \right\|_* \end{aligned}$$

$$\begin{aligned} & \leq \left\| \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} F_0(q_3 \tilde{\theta}(t)) dt \right\|_* \\ & + \left\| \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} F_0(q_3 \tilde{\theta}(t_k) - q_3 \tilde{\theta}(t)) dt \right\|_* \\ & \leq K_0 |e(\bar{t}_k + \delta_0)|_2 \\ & + K_0 |e(\bar{t}_k)|_2 + (k_3 + k_4) \sqrt{\delta_0} \left\{ \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} \|e(t)\|^2 dt \right\}^{1/2} \\ & + (\hat{\mu}_{d,\max} + \mu_{d,\max}) \delta_0 \\ & + k_5 \left[2k_5 \int_{\bar{t}_k}^{\bar{t}_k + \tau_0 + \delta_0} \|e(t)\|^2 dt \right. \\ & + 2k_5 (\|v(t)\|_{L_\infty(0,\infty;Y)} + \|g_0\|_{L_\infty(0,\infty;Y)}) \\ & \times \sqrt{\tau_0 + \delta_0} \left\{ \int_{\bar{t}_k}^{\bar{t}_k + \tau_0 + \delta_0} \|e(t)\|^2 dt \right\}^{1/2} \\ & + \left. \frac{1}{\rho} \int_{\bar{t}_k}^{\bar{t}_k + \tau_0 + \delta_0} | \langle -\delta \hat{\theta} + \gamma_0 |e + e_t|_0 + \gamma g - \dot{\theta}, \tilde{\theta}/\gamma \rangle_{\mathcal{Q}} | dt \right] \\ & \times \left[\sqrt{\delta_0} \left\{ \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} \|e(t)\|^2 dt \right\}^{1/2} \right. \\ & \left. + \delta_0 (\|v(t)\|_{L_\infty(0,\infty;Y)} + \|g_0\|_{L_\infty(0,\infty;Y)}) \right]. \end{aligned} \tag{62}$$

From the adaptive laws (22.a-c) and (45.b) we can have

$$\langle -\delta \hat{\theta} + \gamma g - \dot{\theta}, \tilde{\theta}/\gamma \rangle_{\mathcal{Q}} \leq v'(t),$$

$$\text{where } v'(t) = \sum_{i=1}^3 \left(\varepsilon_i + \xi_i \|\theta_i\|_{\mathcal{Q}} + \frac{\delta_i}{\gamma_i} \|\theta_i\|_{\mathcal{Q}}^2 + \frac{1}{\gamma} \langle \dot{\theta}_i, \theta_i \rangle_{\mathcal{Q}} \right)$$

and $v'(t)$ can be made arbitrary small near to zero by making sufficiently small ε_i 's, δ_i 's, and sufficiently large γ_i 's.

Now, from Appendix A, for any $L > 0$, $\lim_{t \rightarrow \infty} \int_t^{t+L} \|e(s)\|^2 ds \approx 0$. Therefore, letting $k \rightarrow \infty$ in (62) and sufficiently small ε_i 's, ε_d , δ_i 's, δ_d and sufficiently large γ_i 's, γ_d , Theorem 1 and Appendix A imply that

$$\begin{aligned} & 0 < c_2 \varepsilon_0 \\ & \leq K_0 \lim_{k \rightarrow \infty} |e(\bar{t}_k + \delta_0)|_2 + K_0 \lim_{k \rightarrow \infty} |e(\bar{t}_k)|_2 \\ & + (k_3 + k_4) \sqrt{\delta_0} \left\{ \lim_{k \rightarrow \infty} \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} \|e(t)\|^2 dt \right\}^{1/2} \\ & + k_5 \left[2k_5 \lim_{k \rightarrow \infty} \int_{\bar{t}_k}^{\bar{t}_k + \tau_0 + \delta_0} \|e(t)\|^2 dt \right. \\ & + 2k_5 (\|v(t)\|_{L_\infty(0,\infty;Y)} + \|g_0\|_{L_\infty(0,\infty;Y)}) \\ & \times \sqrt{\tau_0 + \delta_0} \left\{ \lim_{k \rightarrow \infty} \int_{\bar{t}_k}^{\bar{t}_k + \tau_0 + \delta_0} \|e(t)\|^2 dt \right\}^{1/2} \\ & + \left. \frac{1}{\rho} \lim_{k \rightarrow \infty} \int_{\bar{t}_k}^{\bar{t}_k + \tau_0 + \delta_0} | \langle -\delta \hat{\theta} + \gamma_0 |e + e_t|_0 + \gamma g - \dot{\theta}, \tilde{\theta}/\gamma \rangle_{\mathcal{Q}} | dt \right] \\ & \times \left[\sqrt{\delta_0} \left\{ \lim_{k \rightarrow \infty} \int_{\bar{t}_k}^{\bar{t}_k + \delta_0} \|e(t)\|^2 dt \right\}^{1/2} \right. \\ & \left. + \delta_0 (\|v(t)\|_{L_\infty(0,\infty;Y)} + \|g_0\|_{L_\infty(0,\infty;Y)}) \right] \end{aligned}$$

≈ 0 ,

which is a contradiction, and the theorem is proved.

5. Conclusions

A robust MRAC algorithm for a cantilevered flexible structure with unknown spatiotemporally varying coefficients and disturbance has been investigated in this paper. The spatiotemporally varying coefficients were assumed to be uniformly bounded with uniformly bounded derivatives, but they were allowed to vary arbitrarily fast. The disturbance was also assumed to be uniformly bounded. Under the unknown plant parameters and external disturbances, the robust MRAC law proposed assures the closed loop system to track a desired signal that comes from the reference model. Because the derivative of a Lyapunov function candidate was not negative semidefinite, only uniform ultimate boundedness would have been concluded. However, further analysis in this paper has shown that the state error, which remains in the derivative of the Lyapunov function candidate, converges near to zero. Also, with the additional persistence of excitation condition, the algorithm guaranteed the convergence of the adjustable controller parameters near to their nominal values. The feasibility of using a finite number of sensors and actuators is under investigation.

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Appendix A: Tracking Error Convergence

From (41), (46), and (47), there exists a positive constant β_0 such that the following holds:

$$\begin{aligned} \dot{V} &\leq -\langle \mu e, e \rangle - \langle (\mu_0 - 1)e_t, e_t \rangle - \sigma_1(q_1^*; e, e) - \sigma_2(q_2^*; e_t, e_t) \\ &\quad - \frac{2\delta_d}{\gamma_d} \mu_d^2 - \sum_{i=1}^3 \frac{\delta_i}{\gamma_i} \|\tilde{\theta}_{i1}\|_Q^2 + \nu(t) \\ &\leq -\mu \|e\|^2 - (\mu_0 - 1) \|e_t\|^2 - k_2(q_1^*) \|e\|_V^2 - k_2(q_2^*) \|e_t\|_V^2 + \nu(t) \\ &\leq -\beta_0 \|e\|_Y^2 + \nu(t), \end{aligned} \quad (\text{A.1})$$

where

$$\begin{aligned} \nu(t) &= 2\varepsilon_d + \frac{\delta_d}{2\gamma_d} \mu_d + \xi_d |\mu_d| + \frac{\delta_d}{\gamma_d} \mu_d^2 + \frac{1}{\gamma_d} \langle \dot{\mu}_d, \mu_d \rangle_Q \\ &\quad + \sum_{i=1}^3 \left(\varepsilon_i + \xi_i \|\theta_i\|_Q + \frac{\delta_i}{\gamma_i} \|\theta_{i1}\|_Q^2 + \frac{1}{\gamma_i} \langle \dot{\theta}_i, \theta_i \rangle_Q \right). \end{aligned}$$

$\nu(t)$ can be made arbitrary small near to zero by making sufficiently small ε_i 's, ε_d , δ_i 's, δ_d and sufficiently large γ_i 's, γ_d . Thus, from Theorem 1 the following is satisfied:

$$\lim_{t \rightarrow \infty} \int_t^{t+L} \|e(s)\|_Y^2 ds \approx 0 \quad \text{for any } L > 0.$$

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